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A FREE BOUNDARY PROBLEM ARISING FROM
A BISTABLE REACTION-DIFFUSION EQUATION

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SIGNIFICANCE AND EXPLANATION

The mathematical equation studied here has been considered as a model for population genetics, combustion, and nerve conduction. A common feature to all of these phenomena is the existence of traveling wave solutions. These may correspond, for example, to the spread of an advantageous gene through a population or the propagation of electrical impulses in a nerve axon. Another common feature is the existence of a threshold phenomenon. In the nerve, for example, a small initial stimulus will not trigger an impulse. If the initial stimulus is greater than some threshold amount, however, a signal will propagate down the axon. In this case the signal quickly assumes a fixed shape and travels with constant velocity. Physiologically, it has been demonstrated that this shape and velocity is independent of the initial stimulus, as long as the stimulus is above threshold.

In this report we demonstrate that the mathematical model under consideration does indeed exhibit a threshold phenomenon. We also study how initial stimuli evolve into traveling wave solutions.

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A FREE BOUNDARY PROBLEM ARISING FROM A BISTABLE REACTION-DIFFUSION EQUATION

David Terman

Section 1. Introduction

In this paper we consider the pure initial value problem for the equation

$$(1.1) \quad v_t = v_{xx} + f(v), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+;$$

the initial datum being $v(x, 0) = \varphi(x)$. We assume that $f(v) = v - H(v - a)$ where H is the Heaviside step function, and $a \in (0, \frac{1}{2})$. This equation, but with smooth f , has many applications and has been studied by a number of authors (see [1], [3], [6]). Equation (1.1) is also a special case of the FitzHugh-Nagumo equations:

$$(1.2) \quad \begin{aligned} v_t &= v_{xx} + f(v) - w \\ w_t &= \varepsilon(v - \gamma w), \quad \varepsilon > 0, \gamma > 0, \end{aligned}$$

which were introduced as a model for the conduction of electrical impulses in the nerve axon. Note that (1.1) can be obtained from (1.2) by setting $\varepsilon = 0$ and $w \equiv 0$ in $\mathbb{R} \times \mathbb{R}^+$. In their original model, FitzHugh [4] and Nagumo, et al., [8] chose $f(v) = v(1 - v)(v - a)$. McKean [7] suggested the further simplification $f(v) = v - H(v - a)$. The results of this paper will be needed in a forthcoming paper when we treat the full system (1.2).

Our primary interest is to study the threshold properties of equation (1.1). That is, if the initial datum $\varphi(x)$ is sufficiently small then one expects the solution of equation (1.1) to decay exponentially fast to zero as $t \rightarrow \infty$. This corresponds to the biological fact that a minimum stimulus is needed to trigger a nerve impulse. In this case we say that $\varphi(x)$ is subthreshold. One expects, however, that if $\varphi(x)$ is sufficiently large, or superthreshold, then some sort of signal will propagate. Threshold results for equation (1.1) with smooth f have been given by Aronson and Weinberger [1]. Fife and McLeod [3] showed that if the initial datum is super-

threshold, then the solution of equation (1.1), with smooth f , will converge to a traveling wave solution.

Throughout this paper we assume that the initial datum, $\varphi(x)$, satisfies the following conditions:

- (a) $\varphi(x) \in C^1(\mathbb{R})$,
- (b) $\varphi(x) \in [0,1]$ in \mathbb{R} ,
- (c) $\varphi(x) = \varphi(-x)$ in \mathbb{R} ,
- (1.3) (d) $\varphi'(x) < 0$ in \mathbb{R}^+ ,
- (e) $\varphi(x_0) = a$ for some $x_0 > 0$,
- (f) $\varphi''(x)$ is a bounded, continuous function except possibly at $|x| = x_0$.

This last condition is needed in order to obtain sufficient a priori bounds on the derivatives of the solution of equation (1.1).

Note that in some sense x_0 determines the size of the initial datum. We expect, therefore, a signal to propagate if x_0 is sufficiently large. In order to be more precise we consider the curve $s(t)$ given by

$$(1.4) \quad s(t) = \sup\{x: v(x,t) = a\}.$$

We say that the initial datum is superthreshold if $s(t)$ is defined in \mathbb{R}^+ and

$\lim_{t \rightarrow \infty} s(t) = +\infty$. In this paper we show that if x_0 is sufficiently large then $\varphi(x)$ is indeed superthreshold.

Note that because $f(v)$ is discontinuous we cannot expect the solution of equation (1.1) to be very smooth. By a classical solution of equation (1.1) we mean the following:

Definition: Let $S_T = \mathbb{R} \times (0,T)$ and $G_T = \{(x,t) \in S_T, v(x,t) \neq a\}$. Then $v(x,t)$ is said to be a classical solution of the Cauchy problem (1.1) in S_T if

- (a) v , along with v_x , are bounded continuous functions in S_T ,
- (b) in G_T , v_{xx} and v_t are continuous functions which satisfy the equation

$$v_t = v_{xx} + f(v)$$

- (c) $\lim_{t \rightarrow 0} v(x,t) = \varphi(x)$ for each $x \in \mathbb{R}$.

We can now state our primary result.

Theorem 1.1: Choose $a \in (0, \frac{1}{2})$. Then there exists a positive constant θ such that if $\varphi(x)$ satisfies the conditions (1.3) with $x_0 > \theta$, then equation (1.1) possesses a classical solution in $\mathbb{R} \times \mathbb{R}^+$, and $\varphi(x)$ is superthreshold. Furthermore, $s(t) \in C^1(\mathbb{R}^+)$, and $s'(t)$ is a locally Lipschitz continuous function.

Note that for the model we are considering it is trivial to give sufficient conditions for the initial datum to be subthreshold. In particular, if $\varphi(x) < a$ for each $x \in \mathbb{R}$ then, from the maximum principle (see [9], page 159), $v(x, t) < a$ in $\mathbb{R} \times \mathbb{R}^+$. Hence v satisfies the equation

$$v_t = v_{xx} - v \text{ in } \mathbb{R} \times \mathbb{R}^+.$$

From this it follows that $\|v(\cdot, t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$, and the initial datum is subthreshold.

We prove Theorem 1.1 by studying the curve $s(t)$ given by (1.4). Note that if the initial datum $\varphi(x)$ satisfies the conditions (1.3) then there must exist some positive time T such that in the interval $[0, T]$, $s(t)$ satisfies the integral equation

$$(1.5) \quad a - \int_{-\infty}^{\infty} K(s(t) - \xi, t) \varphi(\xi) d\xi = \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(s(t) - \xi, t - \tau) d\xi$$

where $K(x, t) = \frac{e^{-t}}{2\pi^{1/2} t^{1/2}} e^{-x^2/4t}$ is the fundamental solution of the linear differential equation $\psi_t = \psi_{xx} - \psi$. Here we give a formal explanation of why this is true. We then show how to construct a solution of the initial value problem (1.1) given a smooth solution of the integral equation (1.5).

From assumptions 1.3(c) and (d) we expect that $v_x(x, t) < 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$. In this case $s(t)$ will be a well defined, continuous function for some time, say $t \in [0, T]$.

It also follows that $v > a$ for $|x| < s(t)$ and $v < a$ for $|x| > s(t)$. Let χ_G be the indicator function of the set $G = \{(x,t): v(x,t) > a; 0 < t < T\}$. Then, for $|x| \neq s(t)$, $v(x,t)$ satisfies the inhomogeneous equation

$$(1.6) \quad v_t = v_{xx} - v + \chi_G$$

with initial datum $v(x,0) = \varphi(x)$. Formally the solution of (1.6) can be written as

$$(1.7) \quad v(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(x-\xi,t-\tau)d\xi.$$

Setting $x = s(t)$ in (1.7) we obtain (1.5).

Lemma 1.2: Suppose that $s(t)$ is a continuously differentiable function which satisfies the integral equation (1.5) in $[0,T]$. Then the function $v(x,t)$ given by (1.7) is a classical solution of the initial value problem (1.1) in $R \times [0,T]$.

Proof: Setting $x = s(t)$ in equation (1.7) and subtracting the resulting equation from (1.5) we find that $v(s(t),t) = a$ in $[0,T]$. Differentiating both sides of (1.7) we see that for $x \neq s(t)$, $v(x,t)$ satisfies the differential equation $v_t = v_{xx} + f(v)$ in $R \times (0,T]$. It also follows from (1.7) that $\lim_{t \rightarrow 0} v(x,t) = \varphi(x)$ for $x \in R$. We now show that $v(x,t)$ is differentiable whenever $x = s(t)$.

First assume that $|\xi| < s(\tau)$. Then $v(\xi,\tau)$ satisfies the differential equation

$$v_\tau - v_{\xi\xi} + v = 1.$$

Multiplying both sides of this equation by $K(x-\xi,t-\tau)$ and using the fact that

$$K_\tau + K_{\xi\xi} - K = 0 \quad \text{we find that}$$

$$(Kv)_\tau - (Kv_\xi)_\xi + (K_\xi v)_\xi = K.$$

Assuming that $|x| < s(t)$ we integrate this last equation for $-s(\tau) < \xi < s(\tau)$,

$\epsilon < \tau < t - \epsilon$, and let $\epsilon \rightarrow 0$ to obtain:

$$\begin{aligned}
(1.8a) \quad v(x, t) &= \int_{-x_0}^{x_0} K(x - \xi, t) \varphi(\xi) d\xi - \int_0^t K(x - s(\tau), t - \tau) a s'(\tau) d\tau \\
&- \int_0^t K(x + s(\tau), t - \tau) a s'(\tau) d\tau - \int_0^t K(x - s(\tau), t - \tau) v_\xi(s(\tau)^-, \tau) d\tau \\
&+ \int_0^t K(x + s(\tau), t - \tau) v_\xi(-s(\tau)^+, \tau) d\tau + \int_0^t a K_\xi(x - s(\tau), t - \tau) d\tau \\
&- \int_0^t a K_\xi(x + s(\tau), t - \tau) d\tau = \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(x - \xi, t - \tau) d\xi.
\end{aligned}$$

Next assume that $\xi > s(\tau)$. Then $v(\xi, \tau)$ satisfies the differential equation:

$v_\tau - v_{\xi\xi} + v = 0$. Multiplying both sides of this equation by $K(x - \xi, t - \tau)$ we find that

$$(Kv)_\tau - (Kv)_\xi + (K_\xi v)_\xi = 0.$$

We integrate this equation for $s(\tau) < \xi < \infty$, $\varepsilon < \tau < t - \varepsilon$ and let $\varepsilon \rightarrow 0$ to obtain

$$\begin{aligned}
(1.8b) \quad & - \int_{-x_0}^{\infty} K(x - \xi, t) \varphi(\xi) d\xi + \int_0^t K(x - s(\tau), t - \tau) a s'(\tau) d\tau \\
& + \int_0^t K(x - s(\tau), t - \tau) v_\xi(s(\tau)^+, \tau) d\tau - \int_0^t a K_\xi(x - s(\tau), t - \tau) d\tau = 0.
\end{aligned}$$

Similarly, for $\xi < s(\tau)$ we obtain

$$\begin{aligned}
(1.8c) \quad & - \int_{-\infty}^{-x_0} K(x - \xi, t - \tau) d\xi + \int_0^t K(x + s(\tau), t - \tau) a s'(\tau) d\tau \\
& - \int_0^t K(x + s(\tau), t - \tau) v_\xi(-s(\tau)^-, \tau) d\tau + \int_0^t a K_\xi(x + s(\tau), t - \tau) d\tau = 0.
\end{aligned}$$

Adding (1.8a), (1.8b), and (1.8c), and using (1.7) we find that for $t \in (0, T)$

$$(1.9) \quad \int_0^t [K(x - s(\tau), t - \tau) [v_{\xi}(s(\tau)^+, \tau) - v_{\xi}(s(\tau)^-, \tau)] \\ + K(x + s(\tau), t - \tau) [v_{\xi}(-s(\tau)^+, \tau) - v_{\xi}(-s(\tau)^-, \tau)]] d\tau = 0 .$$

However, because of assumption (1.3c) it follows from equation (1.7) that

$v(x, t) = v(-x, t)$ in $\mathbb{R} \times (0, T)$. Therefore, (1.9) can be rewritten as

$$\int_0^t [K(x - s(\tau), t - \tau) - K(x + s(\tau), t - \tau)] [v_{\xi}(s(\tau)^+, \tau) - v_{\xi}(s(\tau)^-, \tau)] d\tau = 0 .$$

From this it follows that $v_x(s(t)^-, t) = v_x(s(t)^+, t)$ for each $t \in (0, T)$. ///

In Section 2 we present some notation and prove a few preliminary results which are needed throughout the rest of the paper. In Section 3 we show that for some time T there exists a solution of the integral equation (1.5) in $[0, T]$. In Section 4 we prove that the solution of (1.5) is unique among Lipschitz continuous functions, and in Section 5 we prove that the solution of (1.5) is continuously differentiable. In fact we show that $s'(t)$ is locally Lipschitz continuous. Finally, in Section 6 we demonstrate that if x_0 is sufficiently large, then the initial datum, $\varphi(x)$, is superthreshold.

Section 2. The Operators Φ and Θ

We first introduce the following notation.

Throughout this paper we assume that $\psi(x, t)$ is the solution of the linear differential equation:

$$(2.1) \quad \psi_t = \psi_{xx} - \psi$$

in $\mathbb{R} \times \mathbb{R}^+$ with initial conditions

$$\psi(x, 0) = \varphi(x) .$$

Note that $\psi(x, t) = \int_{-\infty}^{\infty} K(x - \xi, t) \varphi(\xi) d\xi$ where $K(x, t) = \frac{e^{-t}}{2\pi^{1/2} t^{1/2}} e^{-x^2/4t}$.

Now suppose that $\alpha(t)$ is a positive, continuous function defined for $t \in [0, T]$.

For values of t_0 and t which satisfy $0 < t_0 < t < T$ we define the operators:

$$\Phi(\alpha)(t) = \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t) - \xi, t - \tau) d\xi ,$$

$$\Phi_{t_0}(\alpha)(t) = \Phi(\alpha)(t) - \Phi(\alpha)(t_0) ,$$

$$\Theta(\alpha)(t) = \alpha - \psi(\alpha(t), t) ,$$

$$\Theta_{t_0}(\alpha)(t) = \Theta(\alpha)(t) - \Theta(\alpha)(t_0) = \psi(\alpha(t_0), t_0) - \psi(\alpha(t_1), t_1) .$$

Note that $s(t)$ is a solution of the integral equations (1.5) for $t \in [0, T]$ if and only if

$$\Theta_{t_0}(s)(t) = \Phi_{t_0}(s)(t)$$

for all values of t_0 and t such that $0 < t_0 < t < T$.

Definition: Suppose that $\alpha(t)$ is a positive uniformly Lipschitz continuous function defined in $[0, T]$. We define $\alpha(t)$ to be a lower solution in $[0, T]$ if

$\Phi(\alpha)(t) > \Theta(\alpha)(t)$ in $[0, T]$. If $\Phi(\alpha)(t) < \Theta(\alpha)(t)$ in $[0, T]$ then $\alpha(t)$ is said to be an upper solution in $[0, T]$.

In Theorem 4.1 it is shown that if $\alpha(t)$ and $\beta(t)$ are respectively lower and upper solutions in $[0, T]$ then $\alpha(t) < \beta(t)$ in $[0, T]$. This implies that the

solution of (1.5) is unique among uniformly Lipschitz functions. We prove threshold results by showing that if x_0 is sufficiently large then some vertical line $l_1(t) = \bar{x}$ is a lower solution in \mathbb{R}^+ . This will imply that $s(t) > \bar{x}$ in \mathbb{R}^+ . Using this preliminary result we then show that $\lim_{t \rightarrow \infty} s(t) = \infty$, and hence the initial datum is superthreshold. In the rest of this section we prove those properties of the operators Θ and Φ which are needed for the proof of Theorem 1.1. We assume throughout this section that $\alpha(t)$ and $\beta(t)$ are positive continuous functions defined on an interval $[0, T]$.

Lemma 2.1: Assume that for $t_0 < t_1$, $\alpha(t_0) < \beta(t_0)$, and $\alpha(t_1) > \beta(t_1)$. Then

$$\Theta_{t_0}(\alpha)(t_1) > \Theta_{t_0}(\beta)(t_1).$$

Proof: Recall that $\Theta_{t_0}(\alpha)(t_1) = \psi(\alpha(t_0), t_0) - \psi(\alpha(t_1), t_1)$ where $\psi(x, t)$ is the solution of the linear differential equation

$$\psi_t = \psi_{xx} - \psi$$

with initial datum $\psi(x, 0) = \varphi(x)$. From assumption (1.3)(d) and the comparison theorem (see [9], page 159) applied to $\psi_x(x, t)$ it follows that $\psi_x(x, t) < 0$ in $\mathbb{R} \times \mathbb{R}^+$. Therefore, $\psi(\alpha(t_0), t_0) > \psi(\beta(t_0), t_0)$ and $\psi(\alpha(t_1), t_1) < \psi(\beta(t_1), t_1)$. From this the proof of the lemma follows immediately. ///

Lemma 2.2: Assume that $\alpha(t) > \beta(t)$ in $[0, t_0]$, $\alpha(t) > \beta(t)$ for some $t \in (0, t_0)$, and $\alpha(t_0) = \beta(t_0)$. Then $\Phi(\alpha)(t_0) > \Phi(\beta)(t_0)$.

Proof: This is an immediate consequence of the definition of Φ . ///

Lemma 2.3: Assume that $\alpha(t) \in C^1(0, T)$. Then $\Phi(\alpha)(t) \in C^1(0, T)$ and

$$(2.2) \quad \begin{aligned} \Phi(\alpha)'(t) = & \int_{-x_0}^{x_0} K(\alpha(t) - \xi, t) d\xi + \int_0^t K(\alpha(t) + \alpha(\tau), t - \tau) [\alpha'(\tau) + \\ & + \alpha'(\tau)] d\tau + \int_0^t K(\alpha(t) - \alpha(\tau), t - \tau) [\alpha'(\tau) - \alpha'(t)] d\tau. \end{aligned}$$

Proof: Note that

$$\begin{aligned}
 \phi(a)'(t) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\phi(a)(t + \epsilon) - \phi(a)(t)] \\
 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int_0^{t+\epsilon} d\tau \int_{-a(\tau)}^{a(\tau)} K(a(t + \epsilon) - \xi, t + \epsilon - \tau) d\xi \right. \\
 &\quad \left. - \int_0^t d\tau \int_{-a(\tau)}^{a(\tau)} K(a(t) - \xi, t - \tau) d\xi \right] \\
 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int_{-\epsilon}^t d\tau \int_{-a(\tau)+a(t)-a(t+\epsilon)}^{a(\tau+\epsilon)+a(t)-a(t+\epsilon)} K(a(t) - \xi, t - \tau) d\xi \right. \\
 &\quad \left. - \int_0^t d\tau \int_{-a(\tau)}^{a(\tau)} K(a(t) - \xi, t - \tau) d\xi \right] \\
 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int_{-\epsilon}^0 d\tau \int_{-a(\tau)+a(t)-a(t+\epsilon)}^{a(\tau+\epsilon)+a(t)-a(t+\epsilon)} K(a(t) - \xi, t - \tau) d\xi \right. \\
 &\quad \left. + \int_0^t d\tau \int_{-a(\tau)+a(t)-a(t+\epsilon)}^{-a(\tau)} K(a(t) - \xi, t - \tau) d\xi \right. \\
 &\quad \left. + \int_0^t d\tau \int_{a(\tau)}^{a(\tau+\epsilon)+a(t)-a(t+\epsilon)} K(a(t) - \xi, t - \tau) d\xi \right].
 \end{aligned}$$

Passing to the limit we obtain (2.2). ///

We conclude this section by finding sufficient conditions on the initial datum for there to exist lower and upper solutions. We assume throughout that the initial datum, $\varphi(x)$, satisfies the conditions (1.3). We first wish to prove that there exist positive constants θ and r such that if $x_0 > \theta$ then for some $\bar{x} \in (x_0 - r, x_0)$ the vertical line $z_1(t) = \bar{x}$ is a lower solution on \mathbb{R}^+ . The proof of this result is broken up into a few lemmas.

Lemma 2.4: Let $\phi(x_0)(t) = \int_0^t d\tau \int_{-x_0}^{x_0} K(x_0 - \xi, t - \tau) d\xi$ and fix $\epsilon \in (0, \frac{1}{2} - a)$. There exists a positive constant $\theta(\epsilon)$ such that if $x_0 > \theta(\epsilon)$, then

$$\phi(x_0)(t) + \phi(x_0)'(t) > a + \epsilon \text{ in } \mathbb{R}^+.$$

Proof: Let $a_\epsilon = a + \epsilon$

$$t_0 = -\log(\frac{1}{2} - a_\epsilon)$$

$$\delta = \min\left(\frac{\frac{1}{2} - a_\epsilon}{2t_0}, \frac{\frac{1}{2} - a_\epsilon}{1 + t_0}\right)$$

and

$$\theta(\epsilon) = \max\left(1, 2t_0 \log \frac{2t_0^{1/2}}{\pi^{1/2} \delta}\right).$$

Assume that $x_0 > \theta(\epsilon)$. The proof will be broken into two steps. First assume that $t \in (0, t_0)$. Then, using 2.2,

$$\begin{aligned} \phi(x_0)(t) + \phi(x_0)'(t) &= \int_0^t d\tau \int_{-x_0}^{x_0} K(x_0 - \xi, t - \tau) d\xi + \int_{-x_0}^{x_0} K(x_0 - \xi, t) d\xi \\ &= \int_0^t d\tau \int_{-\infty}^{x_0} K(x_0 - \xi, t - \tau) d\xi + \int_{-\infty}^{x_0} K(x_0 - \xi, t) d\xi \\ (2.3) \quad &= \left[\int_0^t d\tau \int_{-\infty}^{-x_0} K(x_0 - \xi, t - \tau) d\xi + \int_{-\infty}^{-x_0} K(x_0 - \xi, t) d\xi \right] \\ &= \frac{1}{2} - \left[\int_0^t d\tau \int_{-\infty}^{-x_0} K(x_0 - \xi, t - \tau) d\xi + \int_{-\infty}^{-x_0} K(x_0 - \xi, t) d\xi \right]. \end{aligned}$$

We now show that for $\tau \in [0, t)$

$$(2.4) \quad \int_{-\infty}^{-x_0} K(x_0 - \xi, t - \tau) d\xi < \delta.$$

From this and (2.3) it will follow that for $t \in (0, t_0)$,

$$\phi(x_0)(t) + \phi(x_0)'(t) > \frac{1}{2} - (1 + t)\delta > \frac{1}{2} - (1 + t_0)\delta > a_\epsilon.$$

Now (2.4) is true because for $\tau \in [0, t]$:

$$\begin{aligned} \int_{-\infty}^{-x_0} K(x_0 - \xi, t - \tau) d\xi &= \frac{e^{-(t-\tau)}}{2\pi^{1/2} (t-\tau)^{1/2}} \int_{-\infty}^{-x_0} e^{-\frac{(x_0 - \xi)^2}{4(t-\tau)}} d\xi \\ &< \frac{e^{-(t-\tau)}}{2\pi^{1/2} (t-\tau)^{1/2}} \int_{-\infty}^{-x_0} e^{-\frac{(x_0 - \xi)^2}{4(t-\tau)}} d\xi. \end{aligned}$$

The last inequality is true because $x_0 > \theta(\varepsilon) > 1$. Therefore,

$$\begin{aligned} \int_{-x_0}^{x_0} K(x_0 - \xi, t - \tau) d\xi &< \frac{2(t-\tau)^{1/2}}{\pi^{1/2}} e^{-x_0/2(t-\tau)} e^{-(t-\tau)} \\ &< \frac{2t^{1/2}}{\pi^{1/2}} e^{-x_0/2t} \\ &< \frac{2t_0^{1/2}}{\pi^{1/2}} e^{-\theta(\varepsilon)/2t_0} \\ &< \delta. \end{aligned}$$

Now assume that $t > t_0$. Then,

$$\begin{aligned} \phi(x_0)(t) + \phi(x_0)'(t) &> \phi(x_0)(t_0) = \int_0^{t_0} d\tau \int_{-\infty}^{x_0} K(x_0 - \xi, t_0 - \tau) d\xi \\ &= \int_0^{t_0} d\tau \int_{-\infty}^{-x_0} K(x_0 - \xi, t_0 - \tau) d\xi. \end{aligned}$$

Since

$$\int_0^{t_0} d\tau \int_{-\infty}^{x_0} K(x_0 - \xi, t_0 - \tau) d\xi = \frac{1 - e^{-t_0}}{2},$$

we conclude from (2.4) that

$$\phi(x_0)(t) + \phi(x_0)'(t) > \frac{1 - e^{-t_0}}{2} - \delta t_0 > a_\varepsilon.$$

Lemma 2.5: Fix $\epsilon \in (0, \frac{1}{2} - a)$ and let $\theta = \theta(\epsilon)$. Let $\theta_1 = \theta + (\frac{4a}{\epsilon})^{1/2}$ and

$$h_\epsilon(x) = \begin{cases} a & \text{for } |x| < \theta \\ 0 & \text{for } |x| > \theta_1 \\ a - \frac{\epsilon}{4}(x - \theta)^2 & \text{for } x \in (\theta, \theta_1) \\ a - \frac{\epsilon}{4}(x + \theta)^2 & \text{for } x \in (-\theta_1, -\theta) \end{cases}$$

Assume that: a) $\varphi(x) > h_\epsilon(x)$ for $|x| < x_1$

and b) $\varphi(x) > h_\epsilon(x) = 0$ for $|x| > x_1$.

Then there exists $\bar{x} \in (\theta, \theta_1)$ such that the line $\ell_1(t) = \bar{x}$ is a lower solution in \mathbb{R}^+ .

Proof: Because of our assumptions on $\varphi(x)$ there exists a function $\varphi_1(x)$ such that

- (a) $\varphi_1(x) \in C^\infty(-\infty, \infty)$
- (b) $h_\epsilon(x) < \varphi_1(x) < \varphi(x)$ for $|x| < x_1$
- (c) $h_\epsilon(x) < \varphi_1(x) < \varphi(x)$ for $|x| > x_1$
- (2.5) (d) $\varphi_1'(x) < 0$ for $x > 0$
- (e) $\varphi_1(x) = \varphi_1(-x)$ in \mathbb{R}
- (f) $\varphi_1(x) < a + \frac{\epsilon}{2}$ in \mathbb{R}
- (g) $\varphi_1''(x) > -\frac{\epsilon}{2}$ in \mathbb{R} .

From these assumptions it follows that $\varphi_1(\bar{x}) = a$ for some unique constant $\bar{x} > \theta$. Let $\psi_1(x, t)$ be the solution of (2.1) with initial datum $\varphi_1(x)$. Since $\varphi(x) > \varphi_1(x)$ in \mathbb{R}^+ it follows from the maximum principle that $\psi(x, t) > \psi_1(x, t)$ in $\mathbb{R} \times \mathbb{R}^+$. We show that $a - \psi_1(\bar{x}, t) < \phi(\bar{x})(t)$ for $t \in \mathbb{R}$. From this it follows that $a - \psi(\bar{x}, t) < a - \psi_1(\bar{x}, t) < \phi(\bar{x})(t)$ and hence the line $\ell_1(t)$ is a lower solution in \mathbb{R}^+ .

We wish to show that $a - \psi_1(\bar{x}, t) < \phi(\bar{x})(t)$, or $\psi_1(\bar{x}, t) > a - \phi(\bar{x})(t)$ for $t \in \mathbb{R}^+$. Let $g(x, t) = \varphi_1(x) - \phi(\bar{x})(t)$. We show, using a comparison argument, that $\psi_1(x, t) > g(x, t)$ in $\mathbb{R} \times \mathbb{R}^+$. Since $\varphi_1(\bar{x}) = a$ this certainly implies the desired result.

In order to apply the maximum principle note that

$$g(x, 0) = \varphi_1(x) = \psi_1(x, 0),$$

and

$$g_t - g_{xx} + g = -[\phi(\bar{x})(t) + \phi(\bar{x})'(t)] + \varphi_1(x) - \varphi_1''(x) \\ < -(a+\epsilon) + (a + \frac{\epsilon}{2}) + \frac{\epsilon}{2} = 0 = \psi_{1t} - \psi_{1xx} + \psi_1.$$

In this last calculation we used Lemma 2.4 and assumptions 2.5(f) and (g). From the maximum principle (see [9], page 159) we conclude that $\psi_1(x,t) > g(x,t)$ in $\mathbb{R} \times \mathbb{R}^+$, and the result follows. ///

Lemma 2.6: There exists positive constants r and θ such that if the initial datum $\varphi(x)$ satisfies (1.3) with $x_0 > \theta$, then, for some $\bar{x} \in (x_0 - r, x_0)$, the line $l_1(t) = \bar{x}$ is a lower solution in \mathbb{R}^+ .

Proof: Choose $\epsilon \in (0, \frac{1}{2} - a)$, $r = (\frac{4a}{\epsilon})^{1/2}$, and $\theta = \theta(\epsilon) + r$. The result now follows from the previous lemma. ///

We now prove the existence of an upper solution.

Lemma 2.7: There exists a linear function $l_2(t)$ such that $l_2(0) = x_0$ and $l_2(t)$ is an upper solution on $[0, \frac{a}{2}]$.

Proof: Recall the function $\psi(x,t)$ defined to be the solution of equation (2.1) with initial datum $\psi(x,0) = \varphi(x)$. From assumptions (1.3d) and (1.3f) it follows that there exist positive constants δ_1 and δ_2 such that $|\psi_t(x,t)| < \delta_2$ and $\psi_x(x,t) < -\delta_1$ in the region $(\frac{x_0}{2}, \infty) \times (0, \frac{a}{2})$. Let $M = \frac{1 + \delta_2}{\delta_1}$, and define $l_2(t)$ by $l_2(t) = Mt + x_0$.

In order to show that $l_2(t)$ is a supersolution in $[0, \frac{a}{2}]$ consider the curve $\beta(t)$ defined implicitly by the equation $\psi(\beta(t), t) = a - t$, $\beta(0) = x_0$. Note that $\beta'(t) = \frac{-1 - \psi_t(\beta(t), t)}{\psi_x(\beta(t), t)} < M$. Hence $\beta(t) < l_2(t)$ in $(0, \frac{a}{2})$. From Lemma 2.1 it follows that for $t \in (0, \frac{a}{2})$,

$$\Theta(l_2)(t) > \Theta(\beta)(t) = a - \psi(\beta(t), t) = t.$$

On the other hand,

$$\Phi(l_2)(t) = \int_0^t d\tau \int_{-l_2(\tau)}^{l_2(\tau)} K(l_2(t) - \xi, t - \tau) d\xi < \int_0^t 1 d\tau = t.$$

Therefore, $\Phi(l_2)(t) < \Theta(l_2)(t)$ for $t \in (0, \frac{a}{2})$, which means that $l_2(t)$ is a supersolution in $[0, \frac{a}{2}]$. ///

Section 3: Existence of $s(t)$

Throughout this section we assume that there exists linear functions $\ell_1(t)$ and $\ell_2(t)$ which are respectively lower and upper solutions in $[0, T]$ for some positive time T . Recall that $s(t)$ is a solution of the integral equation (1.5) in $[0, T]$ if and only if

$$\Phi_{t_0}(s)(t) = \Theta_{t_0}(s)(t)$$

for $0 < t_0 < t < T$. We prove the existence of a solution of (1.5) in $[0, T]$ by constructing a sequence of continuous, piecewise linear functions $\{s_n(t)\}$ with the properties that $s_n(0) = x_0$ and, setting $t_j = \frac{jT}{n}$,

$$\Theta_{t_j}(s_n)(t_{j+1}) = \Phi_{t_j}(s_n)(t_{j+1}) \text{ for } j = 0, \dots, n-1; n = 1, 2, \dots$$

This sequence of functions is shown to be equicontinuous and uniformly bounded.

Therefore, by the theorem of Arzela and Ascoli some subsequence of $\{s_n\}$ converges uniformly to a continuous function. This continuous function is shown to be a solution of the integral equation (1.5).

Lemma 3.1: For each positive integer n there exists a continuous piecewise linear function $s_n(t)$, defined in $[0, T]$, such that $\ell_1(t) < s_n(t) < \ell_2(t)$ and, setting $t_j = \frac{jT}{n}$,

$$\Phi_{t_j}(s_n)(t_{j+1}) = \Theta_{t_j}(s_n)(t_{j+1}), \quad j = 0, 1, \dots, n-1; n = 0, 1, \dots$$

Proof: Fix n . Set $s_n(0) = x_0$ and suppose that we have found points x_0, x_1, \dots, x_k such that $\ell_1(t_j) < x_j < \ell_2(t_j)$, $j = 0, 1, \dots, k$, and, if $s_n(t)$ is the piecewise linear function connecting the points (x_j, t_j) , then

$$\Phi_{t_j}(s_n)(t_{j+1}) = \Theta_{t_j}(s_n)(t_{j+1}); \quad j = 0, 1, \dots, k-1.$$

For $x \in (\ell_1(t_{k+1}), \ell_2(t_{k+1}))$, let

$$\alpha(x)(t) = \begin{cases} s_n(t) & \text{for } t < t_{nk} \\ \text{The line segment connecting } (x_k, t_k) \text{ and} \\ (x, t_{k+1}) & \text{for } t_k < t < t_{k+1}. \end{cases}$$

By induction the proof of the lemma will be complete once we have proven the existence of a point x_{k+1} such that $l_1(t_{k+1}) < x_{k+1} < l_2(t_{k+1})$, and

$\phi_{t_k}(\alpha(x_{k+1}))(t_{k+1}) = \Theta_{t_k}(\alpha(x_{k+1}))(t_{k+1})$. To prove the existence of x_{k+1} we first let $x^1 = l_1(t_{k+1})$ and show that $\phi_{t_k}(\alpha(x^1))(t_{k+1}) - \Theta_{t_k}(\alpha(x^1))(t_{k+1}) > 0$. We then let $x^2 = l_2(t_{k+1})$ and show that $\phi_{t_k}(\alpha(x^2))(t_{k+1}) - \Theta_{t_k}(\alpha(x^2))(t_{k+1}) < 0$. Since $\phi_{t_k}(\alpha(x))(t_{k+1}) - \Theta_{t_k}(\alpha(x))(t_{k+1})$ is a continuous function of x it will then follow that there must exist a point $x_{k+1} \in [x^1, x^2]$ such that

$$\phi_{t_k}(\alpha(x_{k+1}))(t_{k+1}) - \Theta_{t_k}(\alpha(x_{k+1}))(t_{k+1}) = 0.$$

Note that $\alpha(x^1)(t) > l_1(t)$ for $t \in (0, t_{k+1})$. From Lemma 2.2 it follows that $\phi(\alpha(x^1))(t_{k+1}) > \phi(l_1)(t_{k+1})$. From Lemma 2.1 it follows that $\Theta(\alpha(x^1))(t_{k+1}) = \Theta(l_1)(t_{k+1})$. Therefore, since $l_1(t)$ is a lower solution, $\phi(\alpha(x^1))(t_{k+1}) - \Theta(\alpha(x^1))(t_{k+1}) > \phi(l_1)(t_{k+1}) - \Theta(l_1)(t_{k+1}) > 0$. Since $\alpha(x^1)(t) = s_n(t)$ for $t \in (0, t_k)$ it follows that $\phi(\alpha(x^1))(t_k) - \Theta(\alpha(x^1))(t_k) = \phi(s_n)(t_k) - \Theta(s_n)(t_k) = 0$. Hence,

$$\begin{aligned} & \phi_{t_k}(\alpha(x^1))(t_{k+1}) - \Theta_{t_k}(\alpha(x^1))(t_{k+1}) \\ &= [\phi(\alpha(x^1))(t_{k+1}) - \Theta(\alpha(x^1))(t_{k+1})] - [\phi(\alpha(x^1))(t_k) - \Theta(\alpha(x^1))(t_k)] > 0. \end{aligned}$$

A similar argument shows that $\phi_{t_k}(\alpha(x^2))(t_{k+1}) - \Theta_{t_k}(\alpha(x^2))(t_{k+1}) < 0$. From our previous remarks this completes the proof of the lemma. ///

In order to apply the theorem of Arzela and Ascoli to conclude that a subsequence of $\{s_n(t)\}$ converges uniformly to a continuous function we need to show that the sequence $\{s_n(t)\}$ is equicontinuous. We now prove this to be true if T is chosen sufficiently small.

Lemma 3.2: If T is chosen so that $e^{-\frac{l_1(T)}{T}} < \frac{1}{4}$ then the sequence $\{s_n(t)\}$ is equicontinuous on $[0, T]$.

Proof: Let B be the region bounded by $l_1(t)$, $l_2(t)$, $t = 0$ and $t = T$. From assumption (1.3d) it follows that $\psi_x(x, t) < 0$ in B . Choose δ_1 to be a positive constant such that $\psi_x(x, t) < -\delta_1$ in B . From assumption (1.3f) there exists a

positive constant δ_2 such that $|\psi_t(x,t)| < \delta_2$ in B (see [4], Theorem 6, pg. 65).

Let $M = \sup_{0 < \tau < t < T} K(l_1(t) + l_1(\tau), t - \tau)$ and $\bar{t} = \min(\frac{\delta_1}{4M}, T)$.

Since each function $s_n(t)$ is piecewise linear it suffices to show that the derivatives $s'_n(t)$ are uniformly bounded whenever they exist. We first find a lower bound on $s'_n(t)$ for $t \in [0, T]$ and $n = 0, 1, 2, \dots$. In fact, suppose that p is a positive integer such that $p\bar{t} < T$. We show that $s'_n(t) > -2^p \frac{\delta_2}{\delta_1}$ for each n and $t \in (0, p\bar{t})$ such that $s'_n(t)$ is defined.

Suppose that this is not true. Then there must exist positive integers m and n such that $1 < m < p$, $s'_n(\hat{t}) < -2^m \frac{\delta_2}{\delta_1}$ for some $\hat{t} \in ((m-1)\bar{t}, m\bar{t})$, and $s'_n(t) > -2^{m-1} \frac{\delta_2}{\delta_1}$ for $t < (m-1)\bar{t}$. Since $s_n(t)$ is piecewise linear we may assume that for some integer k , $s'_n(t) > -2^m \frac{\delta_2}{\delta_1}$ for $t < t_k \equiv \frac{kT}{n}$, and $s'_n(t) < -2^m \frac{\delta_2}{\delta_1}$ for $t \in (t_k, t_{k+1})$. We show that $\phi(s_n)'(t) - \Theta(s_n)'(t) > 0$ for $t \in (t_k, t_{k+1})$. This immediately leads to a contradiction because $\phi(s_n)(t_k) - \Theta(s_n)(t_k) = \phi(s_n)(t_{k+1}) - \Theta(s_n)(t_{k+1}) = 0$.

We first estimate $\phi(s_n)'(t)$ for $t \in (t_k, t_{k+1})$. Using (2.2) it follows that:

$$\begin{aligned} \phi(s_n)'(t) &> \int_0^{(m-1)t_1} K(s_n(t) + s_n(\tau), t - \tau)(s'_n(\tau) + s'_n(t))d\tau \\ &\quad + \int_0^{(m-1)t_1} K(s_n(t) - s_n(\tau), t - \tau)(s'_n(\tau) - s'_n(t))d\tau \\ &\quad + \int_{(m-1)t_1}^t K(s_n(t) + s_n(\tau), t - \tau)(s'_n(\tau) + s'_n(t))d\tau \\ &\quad + \int_{(m-1)t_1}^t K(s_n(t) - s_n(\tau), t - \tau)(s'_n(\tau) - s'_n(t))d\tau = [I] + [II]. \end{aligned}$$

We show that $[I] > 0$. Recall that for $\tau \in (0, (m-1)t_1)$,

$$s'_n(\tau) > -2^{m-1} \frac{\delta_2}{\delta_1} > s'_n(t). \text{ Hence}$$

$$[I] > \int_0^{(m-1)t_1} \left[2s'_n(t)K(s_n(t) + s_n(\tau), t - \tau) - \left[s'_n(t) + 2^{m-1} \frac{\delta_2}{\delta_1} \right] K(s_n(t) - s_n(\tau), t - \tau) \right] d\tau.$$

The right hand side is positive if for each $\tau < (m-1)t_1$,

$$\begin{aligned} 2s'_n(t) \frac{e^{-(t-\tau)}}{2^{1/2}(t-\tau)^{1/2}} &= \frac{(s_n(t) + s_n(\tau))^2}{4(t-\tau)} \\ &> \left[s'_n(t) + 2^{m-1} \frac{\delta_2}{\delta_1} \right] \frac{e^{-(t-\tau)}}{2^{1/2}(t-\tau)^{1/2}} = \frac{(s_n(t) - s_n(\tau))^2}{4(t-\tau)} \\ \text{or } e^{-\frac{s_n(t)s_n(\tau)}{t-\tau}} &< \frac{s'_n(t) + 2^{m-1} \frac{\delta_2}{\delta_1}}{2s'_n(t)}. \end{aligned}$$

This is true because

$$e^{-\frac{s_n(t)s_n(\tau)}{t-\tau}} < e^{-\frac{l_1(t)^2}{T}} < \frac{1}{4}$$

by assumption, and

$$\frac{s'_n(t) + 2^{m-1} \frac{\delta_2}{\delta_1}}{2s'_n(t)} = \frac{1}{2} + 2^{m-2} \frac{\delta_2}{s'_n(t)\delta_1} > \frac{1}{4}.$$

We have therefore shown that $[I] > 0$. On the other hand,

$$[II] > \int_{(m-1)t_1}^t 2s'_n(t)K(l_1(t) + l_1(\tau), t - \tau) d\tau > 2s'_n(t)t_1^M.$$

Therefore, $\phi(s_n)'(t) > 2Mt_1s'_n(t)$.

We now show that $\Theta(s_n)'(t) < 2M\bar{t}s_n'(t)$ for $t \in (t_k, t_{k+1})$. This is true because

$$\begin{aligned}\Theta(s_n)'(t) &= -\psi_x(s_n(t), t)s_n'(t) - \psi_t(s_n(t), t) \\ &< \delta_1 s_n'(t) + \delta_2 = 4M\bar{t}s_n'(t) + \delta_2 \\ &< 4M\bar{t}s_n'(t) - 2M\bar{t}s_n'(t) = 2M\bar{t}s_n'(t).\end{aligned}$$

We have therefore shown that $\Phi(s_n)'(t) > \Theta(s_n)'(t)$ for $t \in (t_k, t_{k+1})$. As was mentioned earlier this leads to a contradiction. Hence, the uniform lower bound on $s_n'(t)$ follows. Using a similar argument one can obtain a uniform upper bound on $s_n'(t)$. In fact, if P is chosen so that $P\bar{t} < T$ then one can show that $s_n'(t) < 2^P \left(\frac{1 + \delta_2}{\delta_1} \right)$ for each n and $t \in (0, P\bar{t})$ such that $s_n'(t)$ is defined (see [9] for details). From our previous remarks this concludes the proof of the lemma. ///

Since the sequence $\{s_n(t)\}$ is equicontinuous, and uniformly bounded by the lower and upper solutions $l_1(t)$ and $l_2(t)$ on $[0, T]$, the theorem of Arzela and Ascoli guarantees that a subsequence, $\{s_{nk}(t)\}$, converges uniformly on $[0, T]$ to a uniformly Lipschitz function $s(t)$. To simplify notation we write

$$\{s_{nk}(t)\} = \{s_n(t)\}.$$

Lemma 3.4: $s(t)$ is a solution of the integral equation (1.5) in $[0, T]$.

Proof: Let ε be an arbitrary positive constant and choose $t_0 \in [0, T]$. We show that $|\Phi(s)(t_0) - \Theta(s)(t_0)| < \varepsilon$ by estimating, for sufficiently large n , each term of the inequality

$$\begin{aligned}|\Phi(s)(t_0) - \Theta(s)(t_0)| &< |\Phi(s)(t_0) - \Phi(s_n)(t_k)| \\ &\quad + |\Phi(s_n)(t_k) - \Theta(s_n)(t_k)| + |\Theta(s_n)(t_k) - \Theta(s)(t_0)|.\end{aligned}$$

Here k is chosen so that $t_0 \in (t_k, t_{k+1})$.

It follows from the construction of $s_n(t)$ that $|\Phi(s_n)(t_k) - \Theta(s_n)(t_k)| = 0$. Furthermore, because the function $\psi(x, t)$ is uniformly continuous and the sequence of functions $\{s_n(t)\}$ are uniformly Lipschitz continuous, it follows that $|\Theta(s_n)(t_k) - \Theta(s)(t_0)| < \frac{\varepsilon}{2}$ for n sufficiently large. It remains to show that

$|\phi(s)(t_0) - \phi(s_n)(t_0)| < \frac{\varepsilon}{2}$ for n sufficiently large. Setting

$\lambda = s_n(t_k) - s(t_0)$, this is true because

$$\begin{aligned}
 |\phi(s)(t_0) - \phi(s_n)(t_k)| &= \left| \int_0^{t_0} d\tau \int_{-s(\tau)}^{s(\tau)} K(s(t_0) - \xi, t_0 - \tau) d\xi \right. \\
 &\quad \left. - \int_0^{t_k} d\tau \int_{-s_n(\tau)}^{s_n(\tau)} K(s_n(t_k) - \xi, t_k - \tau) d\xi \right| \\
 &= \left| \int_{t_k - t_0}^0 d\tau \int_{-s(\tau + t_0 - t_k) + \lambda}^{s_n(\tau + t_0 - t_k) + \lambda} K(s_n(t_k) - \xi, t_k - \tau) d\xi \right. \\
 &\quad \left. + \int_0^{t_k} d\tau \int_{-s(\tau + t_0 - t_k) + \lambda}^{-s_n(\tau)} K(s_n(t_k) - \xi, t_k - \tau) d\xi \right. \\
 &\quad \left. + \int_0^{t_k} d\tau \int_{s_n(\tau)}^{s(\tau + t_0 - t_k) + \lambda} K(s_n(t_k) - \xi, t_k - \tau) d\xi \right| \\
 &< |t_k - t_0| + 4 \sup_{0 \leq \tau \leq t_k} |s(\tau + t_0 - t_k) - s_n(\tau)| \int_0^{t_k} \frac{d\tau}{2\pi^{1/2} (t_k - \tau)^{1/2}} \\
 &< \frac{\varepsilon}{2}
 \end{aligned}$$

if n is sufficiently large. In the last inequality we used the fact that $s(t)$ is a Lipschitz continuous function and $|t_k - t_0| < \frac{1}{n}$. ///

Section 4. Uniqueness and a Comparison Theorem

We have so far shown that if T is chosen so that there exists linear functions $l_1(t)$ and $l_2(t)$ which are, respectively, lower and upper solutions in $[0, T]$, and $\frac{l_1(T) - l_2(T)}{T} < \frac{1}{4}$, then there exists a uniformly Lipschitz function $s(t)$ which satisfies (1.5) in $[0, T]$. The following Theorem demonstrates that the solution of (1.5) is unique among uniformly Lipschitz functions.

Theorem 4.1: Suppose that $\alpha(t)$ and $\beta(t)$ are respectively lower and upper solutions in $[0, T_1]$. Then $\alpha(t) < \beta(t)$ in $[0, T]$.

Proof: Note that we must have $\alpha(0) < x_0 < \beta(0)$. If, for example, $\alpha(0) > x_0$, then $\psi(\alpha(0), 0) < a$. It follows there must exist some time, t_0 , such that $\psi(\alpha(t), t) < a - t$ for $t \in (0, t_0)$. Therefore, $\Theta(\alpha)(t) = a - \psi(\alpha(t), t) > t$ for $t \in (0, t_0)$. On the other hand,

$$\begin{aligned} \Phi(\alpha)(t) &= \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(\alpha(t) - \xi, t - \tau) d\xi \\ &< \int_0^t 1 d\tau = t \end{aligned}$$

for all $t \in \mathbb{R}^+$. Hence, $\Theta(\alpha)(t) > \Phi(\alpha)(t)$ in $(0, t_0)$, which contradicts the assumption that $\alpha(t)$ is a subsolution. A similar argument shows that it is impossible for $\beta(0) < x_0$.

If $\alpha(0) < \beta(0)$, then we must have $\alpha(t) < \beta(t)$ in $(0, T)$. If not, we let $t_0 = \inf\{t: \alpha(t) > \beta(t)\}$. Then $\alpha(t_0) = \beta(t_0)$ and $\alpha(t) < \beta(t)$ in $(0, t_0)$. Lemma 2.1 now implies that $\Theta(\alpha)(t_0) > \Theta(\beta)(t_0)$, while Lemma 2.2 implies that $\Phi(\alpha)(t_0) < \Phi(\beta)(t_0)$. Since $\alpha(t)$ is a subsolution and $\beta(t)$ a supersolution, we now have

$$\Theta(\alpha)(t_0) < \Phi(\alpha)(t_0) < \Phi(\beta)(t_0) < \Theta(\beta)(t_0) < \Theta(\alpha)(t_0).$$

This is an obvious contradiction.

Throughout the rest of the proof we assume that $\alpha(0) = \beta(0) = x_0$.

Suppose the lemma is not true, and let $t_0 = \inf\{t: \alpha(t) > \beta(t)\}$. Then, $\alpha(t) = \beta(t)$ for $t \in [0, t_0]$. This is because, if $\alpha(t) < \beta(t)$ for some

$t \in [0, t_0]$, it would follow from Lemmas 2.1 and 2.2 that

$$\Phi(\alpha)(t_0) < \Phi(\beta)(t_0) < \Theta(\beta)(t_0) = \Theta(\alpha)(t_0).$$

This, however, contradicts the assumption that $\alpha(t)$ is a lower solution.

We prove the lemma by showing that there exists some $t > t_0$ such that

$\alpha(t) > \beta(t)$ and $\Phi(\alpha)(t) < \Phi(\beta)(t)$. This leads to a contradiction for the following reason. Since $\alpha(t) > \beta(t)$, and $\alpha(0) = \beta(0)$, it follows from Lemma 2.1 that

$\Theta(\alpha)(t) > \Theta(\beta)(t)$. If it is also true that $\Phi(\alpha)(t) < \Phi(\beta)(t)$, then, since $\beta(t)$ is an upper solution, $\Phi(\alpha)(t) < \Phi(\beta)(t) < \Theta(\beta)(t) < \Theta(\alpha)(t)$. This, however, contradicts the assumption that $\alpha(t)$ is a lower solution on $[0, T]$.

For $t > t_0$, let $\epsilon(t) = \alpha(t) - \beta(t)$. Choose $\bar{t} > t_0$ such that $\epsilon(\bar{t}) > 0$ and $\epsilon(t) < \epsilon(\bar{t})$ in $(0, \bar{t})$. Then,

$$\begin{aligned} \Phi(\beta)(\bar{t}) &= \int_0^{\bar{t}} d\tau \int_{-\beta(\tau)}^{\beta(\tau)} K(\beta(\bar{t}) - \xi, \bar{t} - \tau) d\xi \\ &= \int_0^{\bar{t}} d\tau \int_{-\beta(\tau)+\epsilon(\bar{t})}^{\beta(\tau)+\epsilon(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi \\ &= \Phi(\alpha)(\bar{t}) + \left[\int_0^{t_0} d\tau \int_{\alpha(\tau)}^{\beta(\tau)+\epsilon(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi \right. \\ &\quad \left. - \int_0^{t_0} d\tau \int_{-\alpha(\tau)}^{-\beta(\tau)+\epsilon(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi \right] \\ &\quad + \left[\int_{t_0}^{\bar{t}} d\tau \int_{\alpha(\tau)}^{\beta(\tau)+\epsilon(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi \right. \\ &\quad \left. - \int_{t_0}^{\bar{t}} d\tau \int_{-\alpha(\tau)}^{-\beta(\tau)+\epsilon(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi \right] \\ &= \Phi(\alpha)(\bar{t}) + [I] + [II]. \end{aligned}$$

Recall that we wish to choose \bar{t} so that $\Phi(\beta)(\bar{t}) > \Phi(\alpha)(\bar{t})$. Note that $[I] > 0$.

This is because, if $(\xi, \tau) \in (0, \epsilon(\bar{t})) \times (0, t_0)$, then $|\alpha(\bar{t}) - (\beta(\tau) + \xi)|$

$< |\alpha(\bar{t}) + \beta(\tau) - \xi|$, and, therefore, $K(\alpha(\bar{t}) - (\beta(\tau) + \xi), \bar{t} - \tau)$
 $> K(\alpha(\bar{t}) + \beta(\tau) - \xi, \bar{t} - \tau)$.

To complete the proof of the lemma it remains to choose \bar{t} so that [II] > 0 . We rewrite [II] as

$$[II] = \int_{A_1(\bar{t})} \int K(\alpha(\bar{t}) - \xi, \bar{t} - \tau) d\xi d\tau - \int_{A_2(\bar{t})} \int K(\alpha(\bar{t}) - \xi, t - \tau) d\xi d\tau$$

where $A_1(\bar{t}) = \{(\xi, \tau) : t_0 < \tau < \bar{t}, \alpha(\tau) < \xi < \beta(\tau) + \varepsilon(\bar{t})\}$,

$A_2(\bar{t}) = \{(\xi, \tau) : t_0 < \tau < \bar{t}, -\alpha(\tau) < \xi < -\beta(\tau) + \varepsilon(\bar{t})\}$.

Let $\lambda_1(\bar{t}) = \inf_{(\xi, \tau) \in A_1(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau)$,

$\lambda_2(\bar{t}) = \sup_{(\xi, \tau) \in A_2(\bar{t})} K(\alpha(\bar{t}) - \xi, \bar{t} - \tau)$.

Then [II] $> \lambda_1(\bar{t}) \mu(A_1(\bar{t})) - \lambda_2(\bar{t}) \mu(A_2(\bar{t}))$ where μ is Lebesgue measure on \mathbb{R}^2 .

We now show that $\lim_{t \rightarrow t_0} \lambda_1(t) = \infty$ and $\lim_{t \rightarrow t_0} \lambda_2(t) = 0$. The first limit follows because both $\alpha(t)$ and $\beta(t)$ are uniformly Lipschitz continuous. That is, there exists a constant L such that if $t > t_0$ and $\varepsilon(t) > 0$, then

$|\alpha(t) - \xi| < L(t - \tau)$ for all $(\xi, \tau) \in A_1(t)$. Therefore, if $(\xi, \tau) \in A_1(t)$, then

$$\begin{aligned} K(\alpha(t) - \xi, t - \tau) &= \frac{e^{-(t-\tau)}}{2\pi^{1/2}(t-\tau)^{1/2}} e^{-\frac{(\alpha(t) - \xi)^2}{4(t-\tau)}} \\ &> \frac{e^{-(t-\tau)}}{2\pi^{1/2}(t-\tau)^{1/2}} e^{-\frac{L^2}{4}(t-\tau)} \\ &> \frac{e^{-(t-t_0)}}{2\pi^{1/2}(t-t_0)^{1/2}} e^{-\frac{L^2}{4}(t-t_0)}. \end{aligned}$$

Hence $\lambda_1(t) = \inf_{(\xi, \tau) \in A_1(t)} K(\alpha(t) - \xi, t - \tau) \rightarrow \infty$ as $t \rightarrow t_0$.

On the other hand, $\lambda_2(t) \rightarrow 0$ as $t \rightarrow t_0$ for the following reason. If

$(\xi, \tau) \in A_2(t)$, then $\xi < 0$. Hence, $\alpha(t) - \xi > \alpha(t)$. Therefore, for

$(\xi, \tau) \in A_2(t)$,

$$K(\alpha(t) - \xi, t - \tau) \leq \frac{e^{-(t-\tau)}}{2\pi^{1/2}(t-\tau)^{1/2}} e^{-\frac{\alpha(t)^2}{4(t-\tau)}}$$

From this it follows that $\lambda_2(t) = \sup_{(\xi, \tau) \in A_2(t)} K(\alpha(t) - \xi, t - \tau) \rightarrow 0$ as $t \rightarrow t_0$.

Now choose $t_1 > t_0$ so that $\varepsilon(t_1) > 0$, $\varepsilon(t) < \varepsilon(t_1)$ for $t \in (t_0, t_1)$, and $\lambda_1(t) > 4\lambda_2(t)$ for $t \in (t_0, t_1)$. Let $h(t) = \beta(t) + \frac{t}{t_1} \varepsilon(t_1)$. We consider two cases.

Case 1: Suppose there exists $\bar{t} \in (t_0, t_1)$ such that $\alpha(t) < h(t)$ for all $t < \bar{t}$, and $\alpha(\bar{t}) = h(\bar{t})$. Let $B(\bar{t}) = \{(x, t): t_0 < t < \bar{t}; h(t) < x < \beta(t) + \varepsilon(\bar{t})\}$. Then $B(\bar{t}) \subset A_1(\bar{t})$, and $\mu(B(\bar{t})) = \frac{1}{2} \varepsilon(\bar{t}) \bar{t}$. Therefore, $\mu(A_1(\bar{t})) > \frac{1}{2} \varepsilon(\bar{t}) \bar{t}$. On the other hand, $\mu(A_2(\bar{t})) < 2\varepsilon(\bar{t}) \bar{t}$. It now follows that

$$[II] > \lambda_1(\bar{t}) \mu(A_1(\bar{t})) - \lambda_2(\bar{t}) \mu(A_2(\bar{t})) > 4\lambda_2(\bar{t}) \frac{1}{2} \varepsilon(\bar{t}) \bar{t} - \lambda_2(\bar{t}) 2\varepsilon(\bar{t}) \bar{t} = 0.$$

Case 2: Suppose there exists a sequence $\{t_k\}$ such that $t_k \rightarrow t_0$, $\alpha(t_k) > h(t_k)$, and $\varepsilon(t) < \varepsilon(t_k)$ for $t < t_k$.

Let L be a uniform Lipschitz constant for both $\alpha(t)$ and $\beta(t)$. Choose k so that $\lambda_1(t_k) > \frac{8Lt_1}{\varepsilon(t_1)} \lambda_2(t_2)$.

Let

$$\delta_1(t) = -L(t - t_0) + \alpha(t_0) \quad \text{for } t > t_0,$$

$$\delta_2(t) = L(t - t_0) + \alpha(t_0) \quad \text{for } t > t_0,$$

$$Q = \{(x, t) | \delta_2(t) < x < \delta_1(t) + \varepsilon(t_k), t_0 < t\}.$$

Then $A_1(t_k) \supset Q$, and $\mu(Q) = \frac{1}{4L} [\varepsilon(t_k)]^2$. Therefore, $\mu(A_1(t_k)) > \frac{1}{4L} [\varepsilon(t_k)]^2$. As before, $\mu(A_2(t_k)) < 2\varepsilon(t_k) t_k$. Note that $\varepsilon(t_k) > \frac{t_k}{t_1} \varepsilon(t_1)$. This is because $\alpha(t_k) > h(t_k) = \beta(t_k) + \frac{t_k}{t_1} \varepsilon(t_1)$, and hence, $\varepsilon(t_k) = \alpha(t_k) - \beta(t_k) > \frac{t_k}{t_1} \varepsilon(t_1)$.

Letting $\bar{\epsilon} = t_k$ it now follows that

$$\begin{aligned}
 \text{[II]} &> \lambda_1(t_k) \mu(A_1(t_k)) - \lambda_2(t_k) \mu(A_2(t_k)) \\
 &> \frac{8Lt_1}{\epsilon(t_1)} \lambda_2(t_k) \frac{1}{4L} [\epsilon(t_k)]^2 - \lambda_2(t_k) 2\epsilon(t_k) t_k \\
 &= \frac{t_1}{\epsilon(t_1)} \lambda_2(t_k) [\epsilon(t_k)]^2 - 2\epsilon(t_k) \lambda_2(t_k) t_k \\
 &> \frac{t_1}{\epsilon(t_1)} \lambda_2(t_k) \epsilon(t_k) \frac{t_k}{t_1} \epsilon(t_1) - 2\epsilon(t_k) \lambda_2(t_k) t_k \\
 &= 2\epsilon(t_k) \lambda_2(t_k) t_k - 2\epsilon(t_k) \lambda_2(t_k) t_k = 0.
 \end{aligned}$$

Therefore, $\text{[II]} > 0$, and the proof of the lemma is complete. ///

Note that because $f(v)$ is discontinuous we cannot immediately apply the standard comparison theorems to solutions of Equation 1.1. We can, however, prove the following result which is an application of the preceding theorem.

Theorem 4.2: Suppose that the functions $\varphi_1(x)$ and $\varphi_2(x)$ satisfy the conditions (1.3) with $\varphi_1(x) < \varphi_2(x)$ in \mathbb{R} , and $v_1(x,t)$ and $v_2(x,t)$ are the solutions of Equation 1.1 with initial data $\varphi_1(x)$ and $\varphi_2(x)$. Furthermore, suppose that the curves $\sigma_1(t)$ and $\sigma_2(t)$, given by $v_1(\sigma_1(t),t) = a$, $\sigma_1(t) > 0$, and

$v_2(\sigma_2(t),t) = a$, $\sigma_2(t) > 0$, are well defined and continuously differentiable in $[0,T]$. Then, $\sigma_1(t) < \sigma_2(t)$ in $[0,T]$, and $v_1(x,t) < v_2(x,t)$ in $\mathbb{R} \times [0,T]$.

Proof: We first show that $\sigma_1(t) < \sigma_2(t)$ in $[0,T]$. Let $\psi_1(x,t)$ and $\psi_2(x,t)$ be solutions of the linear differential equation

$$\psi_t = \psi_{xx} - \psi$$

with initial data $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Then $\sigma_1(t)$ is a solution of the integral equation

$$(4.1) \quad a - \psi_1(\sigma_1(t),t) = \phi(\sigma_1)(t) \quad \text{in } (0,T),$$

while $\sigma_2(t)$ is a solution of the integral equation

$$a - \psi_2(\sigma_2(t),t) = \phi(\sigma_2)(t) \quad \text{in } (0,T).$$

Because $\varphi_1(x) < \varphi_2(x)$ in \mathbb{R} it follows from the usual comparison theorem for parabolic equations that $\psi_1(x,t) < \psi_2(x,t)$ in $\mathbb{R} \times (0,T)$. Thus,

$$\psi_1(\sigma_1(t),t) < \psi_2(\sigma_1(t),t) \text{ in } (0,T), \text{ and, from (4.1),}$$

$$a - \psi_2(\sigma_1(t),t) < \phi(\sigma_1)(t) \text{ in } (0,T).$$

That is, $\sigma_1(t)$ is a lower solution on $[0,T]$ for Equation 1.1 with initial data $\varphi_2(x)$. From Theorem 4.1 it follows that $\sigma_1(t) < \sigma_2(t)$ in $(0,T)$.

We now show that $v_1(x,t) < v_2(x,t)$ in $\mathbb{R} \times (0,T)$. First assume that $x > \sigma_2(t)$. Then, since $\sigma_1(t) < \sigma_2(t)$, it follows that $v_1(\sigma_2(t),t) < a = v_2(\sigma_2(t),t)$. We also have that $v_1(x,t) < a$ and $v_2(x,t) < a$ for $x > \sigma_2(t)$, $t \in (0,T)$. Therefore, for $x > \sigma_2(t)$, both $v_1(x,t)$ and $v_2(x,t)$ satisfy the linear differential equations

$$v_t = v_{xx} - v.$$

Since $\varphi_1(x) < \varphi_2(x)$ it now follows from the usual comparison theorem for parabolic equations that $v_1(x,t) < v_2(x,t)$ for $x > \sigma_2(t)$, $t \in (0,T)$.

If $\sigma_1(t) < x < \sigma_2(t)$, then $v_1(x,t) < a < v_2(x,t)$. Finally, if $x \in (0, \sigma_1(t))$ then both $v_1(x,t)$ and $v_2(x,t)$ are greater than the parameter a . Thus, they both satisfy the linear differential equation

$$v_t = v_{xx} - v + 1.$$

Since $v_1(\sigma_1(t),t) = a < v_2(\sigma_1(t),t)$ and $\varphi_1(x) < \varphi_2(x)$, it follows that $v_1(x,t) < v_2(x,t)$ for $x \in (0, \sigma_1(t))$, $t \in (0,T)$.

We have now shown that $v_1(x,t) < v_2(x,t)$ in $\mathbb{R}^+ \times (0,T)$. Since $v_k(-x,t) = v_k(x,t)$, $k = 1,2$, the result follows. ///

Section 5. Regularity of $s(t)$

In this section we prove that $s(t) \in C^1(0, T)$ and $s'(t)$ is a locally Lipschitz continuous function. In the previous section we showed that $s(t)$ is a uniformly Lipschitz continuous function. Hence, there exists a positive constant M such that $|s(t_1) - s(t_0)| < M|t_1 - t_0|$ for $t_0, t_1 \in (0, T)$, and $s'(t)$ exists almost everywhere in $[0, T]$. We first prove the following preliminary result.

Theorem 5.1: Assume that t_0 is chosen so that $s'(t_0)$ exists. Then positive constants ε and M_1 can be chosen so that if $|t_1 - t_0| < \varepsilon$, then there exists a Lipschitz continuous function $\alpha(t)$, defined on $[0, t_1 + \varepsilon]$, such that:

- (a) $\alpha(t) = s(t)$ in $[0, t_1]$
- (b) $|s'(t_0) - \alpha'(t_1^+)| < M_1|t_1 - t_0|$
- (c) $\alpha(t)$ is a lower solution in $[0, t_1 + \varepsilon]$.

Proof: For t_1 sufficiently close to t_0 we define the function $\alpha(t)$ as follows.

For $t < t_1$ let $\alpha(t) = s(t)$, and for $t > t_1$ define $\alpha(t)$ implicitly by

$$(5.1) \quad \Theta_{t_1}(\alpha)(t) = \Theta_{t_0}(\alpha)(t_0 + (t - t_1)) + \gamma|t_1 - t_0|(t - t_1)$$

where the constant γ is to be determined. Since $\psi_x(x, t) \neq 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$ the implicit function theorem guarantees the existence of $\alpha(t)$ in a neighborhood of t_1 . Since $s'(t_0)$ exists it also follows that $\alpha'(t_1^+)$ exists. The proof of Theorem 5.1 is now broken up into a few lemmas.

Lemma 5.2: There exist positive constants ε_1 , K_1 , and M_1 such that if $\gamma > K_1$ and $|t_1 - t_0| < \varepsilon_1$, then $0 < s'(t_0) - \alpha'(t_1^+) < M_1|t_1 - t_0|$.

Proof: Note that

$$\begin{aligned} \alpha'(t_1^+) &= \frac{\psi_x(s(t_0), t_0)}{\psi_x(s(t_1), t_1)} s'(t_0) + \frac{\psi_t(s(t_0), t_0) - \psi_t(s(t_1), t_1)}{\psi_x(s(t_1), t_1)} \\ &\quad + \frac{\gamma|t_1 - t_0|}{\psi_x(s(t_1), t_1)}. \end{aligned}$$

The result now follows because $\psi(x, t)$ is an infinitely differentiable function in

$\mathbb{R} \times \mathbb{R}^+$, $\psi_x(s(t), t)$ is negative and bounded away from zero in $(0, T)$, and $s(t)$ is a uniformly Lipschitz continuous function.

Lemma 5.3: Let ε_1 be as in the preceding lemma. There exists a positive constant K_2 such that if $|t_1 - t_0| < \varepsilon_1$ and $\gamma > K_2$, then

$$\phi(\alpha)'(t_1^+) - \phi(s)'(t_0) > -\frac{\gamma}{2} |t_1 - t_0|.$$

Proof: From (2.2) it follows that

$$\begin{aligned} \phi(\alpha)'(t_1^+) - \phi(s)'(t_0) &= \left[\int_{-x_0}^{x_0} K(\alpha(t_1) - \xi, t_1) - K(s(t_0) - \xi, t_0) d\xi \right] \\ &+ \left[\int_0^{t_1} K(\alpha(t_1) - \alpha(\tau), t_1 - \tau) [\alpha'(\tau) - \alpha'(t_1^+)] d\tau \right. \\ &\quad \left. - \int_0^{t_0} K(s(t_0) - s(\tau), t_0 - \tau) [s'(\tau) - s'(t_0)] d\tau \right] \\ &+ \left[\int_0^{t_1} K(\alpha(t_1) + \alpha(\tau), t_1 - \tau) [\alpha'(\tau) + \alpha'(t_1^+)] d\tau \right. \\ &\quad \left. - \int_0^{t_0} K(s(t_0) + s(\tau), t_0 - \tau) [s'(\tau) + s'(t_0)] d\tau \right] \\ &= [A] + [B] + [C]. \end{aligned}$$

Since $K(x, t)$ is infinitely differentiable for $t > 0$, it follows that there exists a positive constant D_1 , independent of γ , such that

$$[A] > -D_1 |t_1 - t_0|.$$

We now consider $[B]$. Assume that $t_1 > t_0$. The case $t_1 < t_0$ is similar. Since $\alpha(t) = s(t)$ for $t < t_1$ we may rewrite $[B]$ as

$$\begin{aligned} [B] &= \int_0^{t_0} [K(s(t_1) - s(\tau), t_1 - \tau) - K(s(t_0) - s(\tau), t_0 - \tau)] s'(\tau) d\tau \\ &+ \int_0^{t_0} [K(s(t_0) - s(\tau), t_0 - \tau) s'(t_0) - K(s(t_1) - s(\tau), t_1 - \tau) \alpha'(t_1^+)] d\tau \\ &+ \int_{t_0}^{t_1} K(s(t_1) - s(\tau), t_1 - \tau) (\alpha'(\tau) - \alpha'(t_1^+)) d\tau. \end{aligned}$$

Recall that $|s'(t)| < M$ wherever $s'(t)$ exists, and, from the preceding lemma,

$\alpha'(t_1^+) < s'(t_0)$. Therefore,

$$\begin{aligned} [B] &> -2M \int_0^t |K(s(t_1) - s(\tau), t_1 - \tau) - K(s(t_0) - s(\tau), t_0 - \tau)| d\tau \\ &\quad - 2M \int_{t_0}^{t_1} K(s(t_1) - s(\tau), t_1 - \tau) d\tau \\ &> -D_2 |t_1 - t_0| \end{aligned}$$

for some positive constant D_2 , independent of γ .

Similarly,

$$[C] > -D_3 |t_1 - t_0|$$

for some constant D_3 independent of γ . In fact, this computation is easier because

$K(\alpha(t_1) + \alpha(\tau), t_1 - \tau)$ and $K(s(t_0) + s(\tau), t_0 - \tau)$ are smooth functions of τ .

Choosing $K_2 = D_1 + D_2 + D_3$ the result follows.

Lemma 5.4: There exists a positive constant ϵ_2 such that if $|t_1 - t_0| < \epsilon_2$ and $\gamma > K_2$, then $\alpha(t)$ is a lower solution on $[0, t_1 + \epsilon_2]$.

Proof: Since $\alpha(t) = s(t)$ on $[0, t_1]$ it follows that

$$\phi(\alpha)(t) = \Theta(\alpha)(t) \text{ on } [0, t_1].$$

It follows from Lemma 5.3 that there exists a positive constant ϵ_2 such that if

$0 < t - t_1 < \epsilon_2$, $|t_1 - t_0| < \epsilon_2$, and $\gamma > K_2$ then

$$\frac{\phi(\alpha)(t) - \phi(\alpha)(t_1)}{t - t_1} - \frac{\phi(s)(t_0 + (t - t_1)) - \phi(s)(t_0)}{t - t_1} > -\gamma |t_1 - t_0|.$$

That is,

$$\phi_{t_1}(\alpha)(t) - \phi_{t_0}(s)(t_0 + (t - t_1)) > -\gamma |t_1 - t_0| (t - t_1).$$

On the other hand, from the definition of $\alpha(t)$,

$$\Theta_{t_1}(\alpha)(t) - \Theta_{t_0}(s)(t_0 + (t - t_1)) = -\gamma |t_1 - t_0| (t - t_1).$$

Since $\Theta_{t_0}(s)(t_0 + (t - t_1)) = \phi_{t_0}(s)(t_0 + (t - t_1))$ it follows that

$$\phi_{t_1}(a)(t) > \theta_{t_1}(a)(t) \text{ on } [t_1, t_1 + \epsilon_2],$$

and, therefore, $a(t)$ is a subsolution on $[0, t_1 + \epsilon_2]$.

This completes the proof of Theorem 5.1. ///

Theorem 5.5: $s(t) \in C^1(0, T)$. Furthermore $s'(t)$ is a locally Lipschitz continuous function.

Proof: Suppose for the moment that t_0 is chosen so that $s'(t_0)$ exists, and let $a(t)$, ϵ , and M_1 be as in Theorem 5.1. From Theorem 4.1 it follows that $a(t) < s(t)$ in $[0, t_1 + \epsilon]$. Therefore, if $s'(t_1)$ exists, then $s'(t_1) > a'(t_1^+) > s'(t_0) - M_1|t_1 - t_0|$. From the proof of Theorem 5.1 we conclude that ϵ and M_1 may be chosen to depend continuously on t_0 . Therefore, we may choose $\epsilon > 0$ such that if $s'(t_0)$ and $s'(t_1)$ both exist, then $s'(t_1) > s'(t_0) - M_1|t_1 - t_0|$, and, switching the roles of t_1 and t_0 , $s'(t_0) > s'(t_1) - M_1|t_1 - t_0|$. This implies that if $s'(t)$ exists for all $t \in (0, T)$, then $s'(t)$ is a locally Lipschitz continuous function. So it remains to prove that $s'(t)$ exists in $(0, T)$.

Let t_0 now be any point in $(0, T)$. Choose ϵ and M_1 such that if $|t_1 - t_0| + |t_2 - t_0| < \epsilon$, and $s'(t_1)$ and $s'(t_2)$ both exist, then $|s'(t_1) - s'(t_2)| < M_1|t_1 - t_2|$. Let $\{t_n\}$, $n = 1, 2, \dots$, be a sequence which satisfies

- a) $s'(t_n)$ exists for each n
- b) $|t_n - t_0| < \frac{1}{M_1} 2^{-(n+1)}$.

Let $P_n = s'(t_n)$. Note that $\{P_n\}$ forms a Cauchy sequence. This is because if $n > m$, then $|t_n - t_m| < \frac{1}{M_1} 2^{-n}$, and, therefore, $|s'(t_n) - s'(t_m)| <$

$< M_1|t_n - t_m| < 2^{-n}$. Hence, $P = \lim_{n \rightarrow \infty} P_n$ exists. We show that $s'(t_0) = P$.

Let $\eta > 0$ be given and choose n so that $2^{-n} < \frac{\eta}{2}$ and $|P - P_n| < \frac{\eta}{2}$. If t is chosen so that $|t - t_0| < \frac{1}{M_1} 2^{-(n+1)}$ and $s'(t)$ exists, then $|t - t_n| < |t - t_0| + |t_0 - t_n| < \frac{1}{M_1} 2^{-n}$ and therefore, $|s'(t) - P_n| < M_1|t - t_n| < 2^{-n}$. Hence, $|s'(t) - P| < |s'(t) - P_n| + |P_n - P| < \eta$. Since $s(t)$ is absolutely continuous this implies that

$$\left| \frac{s(t) - s(t_0)}{t - t_0} - P \right| = \left| \frac{\int_{t_0}^t s'(\tau) d\tau}{t - t_0} - P \right| < \left| \frac{(P + \eta)(t - t_0)}{t - t_0} - P \right| = \eta.$$

Therefore $s'(t_0) = \lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0} = P. \quad ///$

Section 6. Threshold Results

We have so far proven the following result.

Lemma 6.1: Suppose that there exist linear functions $l_1(t)$ and $l_2(t)$ which are respectively lower and upper solutions on $[0, T]$. Furthermore, assume that $T < \frac{a}{2}$ and $e^{-l_1(T)/T} < \frac{1}{4}$. Then there exists a unique continuously differentiable function $s(t)$ which satisfies the integral equation (1.5) in $[0, T]$. Moreover $s'(t)$ is locally Lipschitz continuous in $(0, T)$.

In this section we find sufficient conditions on the initial datum, $\varphi(x)$, for $s(t)$ to exist in \mathbb{R}^+ and $\lim_{t \rightarrow \infty} s(t) = \infty$.

In Section 2 we discussed the existence of lower and upper solutions. It was shown that an upper solution always exists in $[0, \frac{a}{2}]$, and there exist constants θ and r such that if $x_0 > \theta$, then a vertical line $l_1(t) = \bar{x}$, where $\bar{x} > x_0 - r$, is a lower solution on \mathbb{R}^+ . We now show that if $x_0 > \theta$ then $s(t)$ can be extended to $[T, 2T]$. An induction argument can then be used to show that $s(t)$ exists in \mathbb{R}^+ .

By Lemmas 1.2 and 6.1 the solution, $v(x, t)$, of the equation (1.1) exists in $\mathbb{R} \times [0, T]$. To show that $s(t)$ can be extended to the interval $[T, 2T]$ we wish to apply Lemma 6.1 with $\varphi(x)$ replaced by $v(x, T)$. To do this it is necessary to show that $v(x, T)$ satisfies the assumptions (1.3).

Clearly $v(x, T) \in C^1(\mathbb{R})$. Replacing x by $-x$ in equation (1.7) and using the assumption that $\varphi(x) = \varphi(-x)$ it follows that $v(x, T) = v(-x, T)$ in \mathbb{R} . Applying the maximum principle in the regions $|x| < s(t)$ and $|x| > s(t)$ separately it follows that $v(x, T) \in [0, 1]$ in \mathbb{R} . Moreover, since $s'(t)$ is a Lipschitz continuous function it follows from the Schauder estimates (see [4], page 65) applied to the regions $|x| < s(t)$ and $|x| > s(t)$ separately that $v_{xx}(x, T)$ is a bounded continuous function except possibly at $x = s(T)$. Finally, the maximum principle applied to $v_x(x, t)$ implies that $v_x(x, T) < 0$ in \mathbb{R}^+ . We can now apply Lemma 6.1 to conclude that $s(t)$ can be extended to the interval $[T, 2T]$.

This completes the proof that if $x_0 > \theta$, then $s(t)$ exists in \mathbb{R}^+ . Furthermore, $s(t) > x_0 - r$ in \mathbb{R}^+ where r was defined in Lemma 2.6. It remains to

show that there exists a positive constant θ_0 such that if $x_0 > \theta_0$ then

$\lim_{t \rightarrow \infty} s(t) = \infty$. This is done by constructing a particular function $P(x)$ which we show to be superthreshold. We then prove that if x_0 is sufficiently large then $v(x, T) > P(x)$ for some T . From Theorem 4.2 it then follows that $\varphi(x)$ is superthreshold.

In order to define $P(x)$ note that the ordinary differential equation

$$(6.1) \quad P'' + f(P) = 0$$

has the first integral

$$(6.2) \quad \frac{1}{2} (P')^2 + F(P) = k$$

where k is constant and $F(P) \equiv \int_0^P f(u) du$. Choose $K \in (a, 1)$ so that $F(K) = 0$ and suppose that $\beta \in (K, 1)$. Then $F(\beta) > 0$ and $F'(\beta) = f(\beta) > 0$. Define the length

$$b_\beta = \int_0^\beta \{2F(\beta) - 2F(q)\}^{-1/2} dq.$$

For $|x| < b_\beta$ let $P(x)$ be the solution of (6.1) with first integral

$$\frac{1}{2} (P')^2 + F(P) = F(\beta)$$

and which satisfies the condition $P'(0) = 0$. Then $P(x) > 0$ in

$(-b_\beta, b_\beta)$, $P(x) = P(-x)$ and $P(b_\beta) = P(-b_\beta) = 0$. Define $P(x) \equiv 0$ for $|x| > b_\beta$. We now show that $P(x)$ is superthreshold. Our proof follows Aronson and Weinberger [2, Proposition 2.2].

Lemma 6.2: Let $u(x, t)$ be the solution of equation (1.1) with initial datum $P(x)$.

Then $\lim_{t \rightarrow \infty} u(x, t) = 1$ for each $x \in \mathbb{R}$.

Proof: The proof is broken into two parts. We first show that $\lim_{t \rightarrow \infty} v(x, t) = \tau(x)$ uniformly on each bounded interval where $\tau(x)$ is the smallest solution of (6.1) which satisfies the inequality

$$\tau(x) > P(x) \text{ in } \mathbb{R}.$$

We then show that $\tau(x) \equiv 1$.

From the comparison theorems we conclude that $u(x,t) \in [0,1]$ and $u(x,t) > P(x)$ in $\mathbb{R} \times \mathbb{R}^+$. Hence, for any $h > 0$ we have $u(x,h) > u(x,0)$ in \mathbb{R} . From Theorem 4.2 it follows that for any $h > 0$, $u(x,t+h) > u(x,t)$ in $\mathbb{R} \times \mathbb{R}^+$. Therefore, for any x , $u(x,t)$ is a nondecreasing function of t which is bounded above. Therefore the limit $\tau(x)$ exists. Clearly $\tau(x) \in [0,1]$ and $\tau(x) > P(x)$ in \mathbb{R} .

We now show that $\tau(x)$ is a solution of (6.1) in \mathbb{R} . Define $\sigma(t)$ by $u(\sigma(t),t) = a$, $\sigma(t) > 0$. Note that $\sigma(t)$ is a nondecreasing function. Hence $\lim_{t \rightarrow \infty} \sigma(t) = \bar{x}$ exists for some $\bar{x} \in (0, \infty]$.

Note that for arbitrary $\eta > 0$ and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$,

$$(6.3) \quad u(x,t+\eta) = \int_{-\infty}^{\infty} K(x-\xi,t)u(\xi,\eta)d\xi + \int_{\eta}^{t+\eta} d\tau \int_{-\sigma(\tau)}^{\sigma(\tau)} K(x-\xi,t+\eta-\tau)d\xi.$$

By means of the substitution $s = t - \eta$ in the second integral on the right hand side of (6.3), $u(x,t+\eta)$ can be rewritten in the form

$$u(x,t+\eta) = \int_{-\infty}^{\infty} K(x-\xi,t)u(\xi,\eta)d\xi + \int_0^t ds \int_{-\sigma(s+\eta)}^{\sigma(s+\eta)} K(x-\xi,t-s)d\xi.$$

Since $u(\cdot,\eta) \uparrow \tau(\cdot)$ it follows from the monotone convergence theorem that

$$(6.4) \quad \tau(x) = \int_{-\infty}^{\infty} K(x-\xi,t)\tau(\xi)d\xi + \int_0^t ds \int_{-\bar{x}}^{\bar{x}} K(x-\xi,t-s)d\xi$$

for each $x \in \mathbb{R}$.

From this representation we conclude that τ is continuous. Since the convergence of the continuous functions u to τ is monotone it follows from Dini's theorem that $u \rightarrow \tau$ uniformly on bounded intervals. We now show that $\tau(x)$ satisfies the steady state equation (6.1) in \mathbb{R} .

First assume that $|x| < \bar{x}$. We rewrite (6.4) as

$$\tau(x) = \int_{-\infty}^{\infty} K(x-\xi,t)\tau(\xi)d\xi - \int_{-\infty}^{\infty} ds \int_{\mathbb{R} \setminus [-\bar{x}, \bar{x}]} K(x-\xi,t-s)d\xi + 1 - e^{-t}.$$

It then follows that

$$\tau''(x) = \int_{-\infty}^{\infty} K_{xx}(x - \xi, t) \tau(\xi) d\xi - \int_{-\infty}^{\infty} ds \int_{\mathbb{R} \setminus [-\bar{x}, \bar{x}]} K_{xx}(x - \xi, t - s) d\xi$$

and

$$0 = \frac{d}{dt} \tau(x) = \int_{-\infty}^{\infty} K_t(x - \xi, t) \tau(\xi) d\xi - \int_{-\infty}^{\infty} ds \int_{\mathbb{R} \setminus [-\bar{x}, \bar{x}]} K_t(x - \xi, t - s) d\xi + e^{-t}$$

valid for arbitrary $t > 0$. Since $K(x, t)$ is a solution of the differential equation $K_t = K_{xx} - K$ it follows that $\tau'' + f(\tau) = \tau'' - \tau + 1 = 0$ for $|x| < \bar{x}$. A similar argument shows that $\tau'' + f(\tau) = 0$ for $|x| > \bar{x}$.

Now if $q(x)$ is any solution of (6.1) with $q \in [0, 1]$ in \mathbb{R} and $P(x) < q(x)$ in \mathbb{R} then, from Theorem 4.2, it follows that $u(x, t) < q(x)$ for each $x \in \mathbb{R}$. Hence $\tau(x) < q(x)$ so that τ is the smallest solution with these properties.

Having proven that $\tau(x)$ is a solution of the steady state equation (6.1) it remains to show that $\tau \equiv 1$. Suppose that there exists x_1 such that $\gamma = \tau(x_1) < 1$. Then $\tau(x)$ satisfies (6.2) with $k > F(\gamma)$. Hence $\{k - F(q)\}^{-1/2}$ is defined on $[0, \gamma]$. Therefore $\tau(x)$ is implicitly given by

$$x = x_1 \mp \int_{\tau}^{\gamma} [2(k - f(u))]^{-1/2} du$$

where the sign is determined by $\tau'(x_1)$. It follows that $\tau(x)$ becomes zero with $\tau' \neq 0$ at a finite value of x , so that τ cannot be a nonnegative solution $q'' + f(q) = 0$ for all x . This contradiction shows that $\tau(x) = \lim_{t \rightarrow \infty} u(x, t) = 1$ for each $x \in \mathbb{R}$ and hence $P(x)$ is superthreshold. ///

The following result completes the proof of Theorem 1.1.

Theorem 6.3: Choose $a \in (0, \frac{1}{2})$. There exists a constant θ_0 such that if $\varphi(x)$ satisfies (1.3) with $x_0 > \theta_0$ then $\varphi(x)$ is superthreshold.

Proof: Recall the constants θ and r defined in Lemma 2.6 and b_β , $P(x)$, $u(x,t)$ defined in this section. Let $\theta_0 = \max\{\theta, b_\beta + r\}$. We show that $v(x,T) > P(x)$ in \mathbb{R} for some T . Theorem 4.2 then implies that $v(x,t) > u(x,t-T)$ for $x \in \mathbb{R}$, $t > T$. Since $\lim_{t \rightarrow \infty} u(x,t) = 1$ for each $x \in \mathbb{R}$ it then follows that $\varphi(x)$ is superthreshold.

Since $x_0 > \max\{\theta, b_\beta + r\}$, Lemma 2.6 implies that $s(t) > b_\beta$ in \mathbb{R}^+ .

Therefore, $v(x,t) > a$ for $|x| < b_\beta$. From the maximum principle we conclude that if $z(x,t)$ is the solution of the initial-boundary value problem

$$z_t = z_{xx} - z + 1 \text{ for } |x| < b_\beta, t \in \mathbb{R}^+,$$

$$z(x,0) = \varphi(x) \text{ for } |x| < b_\beta,$$

$$z(b_\beta, t) = z(-b_\beta, t) = a \text{ in } \mathbb{R}^+,$$

then $v(x,t) > z(x,t)$ for $|x| < b_\beta$, $t \in \mathbb{R}^+$. From Friedman [5, page 158] it follows that $\lim_{t \rightarrow \infty} z(x,t) = q(x)$ where $q(x)$ is the solution of the steady state equation

$$q'' - q + 1 = 0 \text{ for } |x| < b_\beta,$$

$$q(-b_\beta) = q(b_\beta) = a.$$

Therefore, there exists T such that $v(x,T) > q(x)$ for $|x| < b_\beta$. It is also true, however, that $q(x) > P(x)$ for $|x| < b_\beta$. This is because if γ is chosen so that $P(\gamma) = a$, $\gamma > 0$, then $P(x)$ satisfies the steady state equations

$$P'' - P + 1 = 0 \text{ for } |x| < \gamma$$

with

$$P(-\gamma) = P(\gamma) = a.$$

Since $\gamma < b_\beta$ it follows that $P(x) < q(x)$ for $|x| < \gamma$. On the other hand, if $x \in (\gamma, b_\beta)$ or $x \in (-b_\beta, -\gamma)$ then $P(x) < a < q(x)$.

We have now shown that for $|x| < b_\beta$, $P(x) < q(x) < v(x,T)$. Finally, if $|x| > b_\beta$ then $P(x) = 0 < v(x,T)$. ///

The following results will be needed in a later paper when we study the full system (1.2).

Theorem 6.4: Choose $a \in (0, \frac{1}{2})$ and let θ_0 be as in Theorem 6.3. Suppose that $d < 1$ and $r_1 > 0$. There exists $T > 0$ such that if $\varphi(x)$ satisfies (1.3) with $x_0 > \theta_0$, then $v(x,t) > d$ for $|x| < r_1$, $t > T$.

Proof: Let $P(x)$ be as in Lemma 6.2. It was shown in the proof of Theorem 6.3 that for some time T_1 , $v(x,t) > P(x)$ for $x \in \mathbb{R}$, $t > T_1$. If $u(x,t)$ is the solution of (1.1) with initial datum $P(x)$, then, from Theorem 4.2., it follows that $v(x,t + T_1) > u(x,t)$ in $\mathbb{R} \times \mathbb{R}^+$. Since $\lim_{t \rightarrow \infty} u(x,t) = 1$ for each $x \in \mathbb{R}$, and $u_x(x,T) < 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$, there exists T_2 such that $u(x,t) > d$ for $|x| < r_1$, $t > T_2$. Hence, if $t > T = T_1 + T_2$, and $|x| < r_1$, then $v(x,t) > u(x,t - T_1) > d$. ///

Corollary 6.5: Choose $a \in (0, \frac{1}{2})$ and $K < \frac{1}{2} - a$. Assume that $d < 1 - K$ and $r_1 > 0$. Then there exist constant θ , r , and T such that if $\varphi(x)$ satisfies (1.3) with $x_0 > \theta$, and $v(x,t)$ is solution of the differential equation:

$$\begin{aligned} v_t &= v_{xx} + f(v) - K \text{ in } \mathbb{R} \times \mathbb{R}^+, \\ v(x,0) &= \varphi(x), \end{aligned}$$

then $v(x,t) > d$ for $|x| < r_1$, $t > T$. Furthermore, the curve $s(t)$, given by $v(s(t),t) = a$, $s(0) = x_0$, is a well defined, smooth function, $s'(t)$ is locally Lipschitz continuous, $\lim_{t \rightarrow \infty} s(t) = \infty$, and $s(t) > x_0 - r$ in \mathbb{R}^+ .

Proof: Let $u(x,t) = v(x,t) + K$. Then $u(x,t)$ is the solution of the differential equation:

$$\begin{aligned} u_t &= u_{xx} + f_1(u) \text{ in } \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) &= \varphi(x) + K \text{ in } \mathbb{R}. \end{aligned}$$

Here,

$$f_1(u) = \begin{cases} -u & \text{for } u < a + K \\ 1 - u & \text{for } u > a + K. \end{cases}$$

Since $a + K < \frac{1}{2}$, the result now follows from applying Theorems 1.1 and 6.4 to $u(x,t)$. ///

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Heaviside step function, and $a \in (0, \frac{1}{2})$. It is demonstrated that this equation exhibits a threshold phenomenon. This is done by considering the curve $s(t)$ defined by $s(t) = \sup\{x: v(x, t) = a\}$. It is shown that if $v(x, 0) < a$ for all x , then $\lim_{t \rightarrow \infty} \|v(\cdot, t)\|_{\infty} = 0$. Moreover, if the initial datum is sufficiently smooth and satisfies $v(x, 0) > a$ on a sufficiently long interval, then $s(t)$ is defined in \mathbb{R}^+ , and $\lim_{t \rightarrow \infty} s(t) = \infty$. Regularity and uniqueness results for the interface, $s(t)$, are also presented.

